

Wednesday, Sep. 3, Lecture 3: Power Series I

Last time ...

- Convergence tests for real series $\sum_{n=1}^{\infty} a_n$.
- Direct comparison test:
 $\sum a_n \leq \sum b_n$
 - $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges
 - $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

Two very important series:

Geometric series:

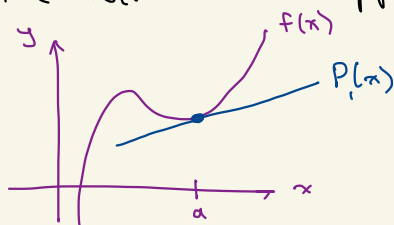
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad ;$$

Exponential series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

Today: Power Series and Taylor Series I

Let's start with the idea of Linear approximation.



- Nearby a , $f(x) \approx P_1(x)$. $P_1(x)$ is a line, so

$$P_1(x) = b + m(x-a)$$

- $P_1(x)$ has the same slope as $f(x)$ at $x=a$, so

$$P_1'(a) = m = f'(a)$$

- $P_1(a) = f(a)$, so we can write

$$P_1(x) = f(a) + f'(a)(x-a)$$

- We can get more accuracy by looking at more derivatives and matching:

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

k^{th} Taylor polynomial centered at a .

- Visually clear that P_k approximates f . But how well?

$$R_k(x) = f(x) - P_k(x)$$

↑

Error

- There isn't a general formula for $R_k(x)$ (if there were, then we could already know $f(x)$, which is pointless). But we can find bounds for its size.

Thm (Taylor's Thm with Remainder)

Fix $a \in \mathbb{R}$.

has any derivative
(actually, just need $k+1$)

Let $k \geq 1$ be an integer and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be smooth*.

Then the Taylor Polynomial $P_k(x)$ approximates $f(x)$ in the following sense:

the error term $R_k(x) = f(x) - P_k(x)$ is bounded by

$$|R_k(x)| \leq M_{k+1} \frac{(x-a)^{k+1}}{(k+1)!}$$

where $M_{k+1} = \max_{y \in [a, x]} |f^{(k+1)}(y)|$.

- In many applications, we know the values of f and its derivatives $f^{(k)}$ at one point.

Ex: Approximating e^x

We want to approximate e^x with a quadratic polynomial on $[-1, 1]$. How good is our approximation?

• Pick $a = 0$. Then

$$e^0 = 1, \quad \left. \frac{d}{dx} e^x \right|_{x=0} = 1, \quad \left. \frac{d^2}{dx^2} e^x \right|_{x=0} = 1$$

$$\Rightarrow P_2(x) = 1 + x + \frac{x^2}{2}.$$

How's the error at the point $x=1$?

$$|e^x - P_2(x)| = |R_2(x)| \leq M \frac{x^3}{3!}$$

$$\text{with } M = \max_{y \in [0, 1]} \left| \left. \frac{d^3}{dx^3} e^x \right|_{x=y} \right| = \max_{y \in [0, 1]} |e^y| = e^1$$

$$\text{so } |R_2(1)| \leq \frac{e(1)^3}{3!} \approx 0.45$$

At worst, $P_2(1)$ is within 0.45 of the true value of e^1 .

So... what if we kept adding terms?

The approximations

$$P_k(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n$$

are a sequence of partial sums...

Def: Taylor series.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be smooth. The Taylor series of f at a is

$$P(x) = \lim_{k \rightarrow \infty} P_k(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

• Note: we know that the error has

$$\lim_{k \rightarrow \infty} |R_k(x)| \leq \lim_{k \rightarrow \infty} M_{k+1} \frac{(x-a)^{k+1}}{(k+1)!}$$

$= 0 \leftarrow$ If M_{k+1} doesn't get too big*. (Like if derivatives are bounded $|f^{(k)}(x)| \leq M$)

Special case: Maclaurin series is $a=0$

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

* Warning: functions are not always equal to their Taylor series (HW 2).

But many important functions are!

$$\bullet e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\bullet \sin x = 0 + x + 0 - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$$

$$\bullet \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

The great news: these are easy to work with!

Let's do one more definition, then put them to work.

Def: Power series

A power series about a is the series of functions.

$$p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Ex: Geometric series

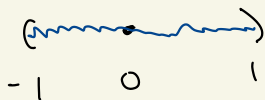
$$p(x) = \sum_{n=0}^{\infty} x^n$$

The radius of convergence R is the largest positive number such that if $|x-a| < R$, then $p(x)$ converges.

$$\left(\begin{array}{ccc} & \bullet & \pi \\ a-R & a & a+R \end{array} \right)$$

Ex:

Geometric series converges for $|x| < 1$ and diverges if $|x| > 1$, so $R=1$:



The great news: when functions are power (Taylor) series, they are very easy to work with!

• We can integrate and differentiate term-by-term:

$$\begin{aligned} \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n &= \frac{d}{dx} (a_0 + a_1 x + a_2 x^2 + \dots) \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \dots \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned}$$

$$\int \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$$

Ex:

A bacterial population is known to reproduce at a rate proportional to the current population:

$$\frac{d}{dt} p = k p$$

What is $p(t)$?

Ansatz: write

$$p(t) = \sum_{n=0}^{\infty} a_n t^n$$

and plug in

$$\frac{d}{dt} (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) = k (a_0 + a_1 t + a_2 t^2 + \dots)$$

$$\Rightarrow a_1 = k a_0 \quad (\text{const})$$

$$2 a_2 = k a_1 \quad (t)$$

$$3 a_3 = k a_2 \quad (t^2)$$

$\vdots$

So, if the initial population is $a_0 = 1$, then

$$a_1 = k$$

$$a_2 = \frac{1}{2} k a_1 = \frac{1}{2} k^2$$

$$a_3 = \frac{1}{3} k a_2 = \frac{1}{3!} k^3$$

$\vdots$

$$a_n = \frac{1}{n!} k^n$$

$$\Rightarrow p(t) = \sum_{n=0}^{\infty} \frac{k^n}{n!} t^n = e^{kt} \quad \text{check: } \frac{d}{dt} (e^{kt}) = k e^{kt} \quad \checkmark$$

$\square$